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Simultaneous approximation on two subsets of an open Riemann surface

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Abstract

Given functions f_1 and f_2 meromorphic, respectively, on subsets E_1 and E_2 of a Riemann surface R, we seek a function meromorphic on all of R, which simultaneously approximates f_1 on E_1 and f_2 on E_2.
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0. Introduction: approximation on a single set

In this paper, R will always denote an open (i.e. a non-compact) Riemann surface, which we always assume to be connected. Topological notions such as closure, interior, etc., will be with respect to R unless otherwise indicated. A subset of R is said to be bounded if its closure is compact. We denote by $R^* = R \cup \{*\}$ the one-point compactification of R, except when R is the complex plane $\mathbb{C}$, in which case we shall use the notation $\overline{\mathbb{C}}$ to denote the Riemann sphere and write ∞ instead of $*$, that is $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. For a subset S of R, let $C(S)$ be the class of continuous complex-valued functions on S, and $H(S)$, respectively, $M(S)$, denotes the space of functions which are holomorphic, respectively, meromorphic, on an open neighbourhood of S. We call $H(R)$ the space of entire functions. Let E be a closed subset of R, and E^{o} denote

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its interior. Then $A(E)$ denotes the space $C(E) \cap H(E^o)$. For a function defined on E, we set $\|f\|_E := \sup\{|f(p)| : p \in E\}$ and we say that f is bounded on E if $\|f\|_E < \infty$. Let $M_b(E)$, respectively, $H_b(E)$, denote the set of functions in $M(R)$, respectively, $H(R)$, which are bounded on E. We call

$$\widehat{E} := \{p \in R : |f(p)| \leqslant \|f\|_E \quad \text{for all } f \in H(R)\}$$

the hull of E, and we say that E is holomorphically convex if $E = \widehat{E}$.

We shall say that a function $f \in M(E)$ can be uniformly approximated on E by functions in $M(R)$, if there exists a sequence $f_n \in M(R)$ such that all f_n have the same set $\mathcal{P}_f$ of poles on E and $|f - f_n|$ converges uniformly to zero on $E \setminus \mathcal{P}_f$.

Let us say that a closed set $E \subset R$ is a meromorphic, respectively, a holomorphic, approximation set, if each function in $M(E)$, respectively, in $H(E)$, can be uniformly approximated on E by functions in $M(R)$, respectively, in $H(R)$. It can be shown that holomorphic approximation sets are meromorphic approximation sets. We shall say that a closed set E is a Roth set if $R^* \setminus E$ is connected and locally connected. A fundamental problem in complex analysis is to describe approximation sets. For surfaces of finite genus, the following theorem gives a complete answer.

Theorem 1 (*Runge theorem, Scheinberg [12,13]*). *Let R be an open Riemann surface of finite genus. Every closed subset E of R is a meromorphic approximation set. Moreover, E is a holomorphic approximation set if and only if it is a Roth set.*

The Roth condition was introduced by Alice Roth. She proved the first part of the above theorem as well as the sufficiency in the second part in 1938 [10] for the case that R is the complex plane $\mathbb{C}$, and in 1973 for arbitrary plane domains [11]. The necessity in this case was proved by N.U. Arakelian [1] in 1964. Arakelian also proved several extensions and refinements of Theorem 1 which he applied to obtain a breakthrough on the inverse problem of value distribution theory, which was later completely solved by Drasin [4].

When E is compact, Theorem 1 is valid on an *arbitrary* open Riemann surface (of finite or infinite genus). When $R = \mathbb{C}$, this is the classical theorem of C. Runge. It was extended to arbitrary open Riemann surfaces by H. Behnke and K. Stein [2] in 1949 . The Behnke–Stein theorem is the most important theorem on open Riemann surfaces and was the starting point for the famous Cartan seminars which contributed so much to the theory of several complex variables.

For closed sets E, the situation on arbitrary open Riemann surfaces is not well understood.

By Theorem 1, on a surface of finite genus, every closed set is a meromorphic approximation set. On a general open Riemann surface, this is not the case (see the following paragraph).

Theorem 1 gives a topological characterization of holomorphic approximation sets when the surface is of finite genus. It is known [7] that on a general Riemann surface, in order for a closed subset E to be a holomorphic approximation set, it remains necessary that E be a Roth set. However, the converse is no longer true. There is an example of a Roth set which is not even a meromorphic approximation set [7]. Thus, the topological characterization of holomorphic approximation sets given by the theorem above no longer holds on general Riemann surfaces. In fact, Scheinberg has shown [12] that *no* topological characterization of holomorphic approximation sets is possible on arbitrary open Riemann surfaces. Even more, he has shown [14] that holomorphic approximation sets are not invariant under a real-analytic isotopy of quasiconformal automorphisms (of a certain R).

0.1. Some meromorphic approximation sets

Let E be a closed subset of an open Riemann surface R. How can we decide whether E is a meromorphic approximation set? First of all, R itself is obviously an approximation set. Henceforth, we assume that $E \neq R$.

If R is of finite genus, then E is always a meromorphic approximation set by Theorem 1.

Consider, more generally, the situation that E has a neighbourhood O, which is a domain of finite genus. Let $\{Q_n\}$ be an exhaustion of R by compact sets, by which we mean: for each n, Q_n is compact, $Q_n \subset Q^{\mathrm{o}}_{n+1}$ and $R = \cup Q^{\mathrm{o}}_n$. It will be convenient to assume that $Q_1 = \emptyset$. For $n = 1, 2, \ldots,$ set $E_n = E \cup Q_n$ and $R_n = O \cup Q^{\mathrm{o}}_{n+2}$. Suppose $f \in M(E)$ and $\varepsilon > 0$. Set $g_0 = f$. By Theorem 1, there is a function $g_1 \in M(R_1)$ such that $|g_1 - g_0| < \varepsilon/2^{-1}$ on E_1. By induction, using Theorem 1, we construct a sequence $g_n \in M(R_n)$ such that $|g_n - g_{n-1}| < \varepsilon/2^{-n}$ on E_n. The sequence $\{g_n\}$ converges to a function $g \in M(R)$ such that $|g - f| < \varepsilon$ on E. Thus E is a meromorphic approximation set.

A set E is said to be *essentially of finite genus* (see [6]) if there exists a covering of E by pairwise disjoint domains $O_j \subset R$, each of which is of finite genus. If E is essentially of finite genus, then E is a set of meromorphic approximation. Indeed, let $\{O_j\}$ be a cover of E by disjoint domains of finite genus. If the collection $\{O_j\}$ is finite, then, it is easy to join them and form a single domain of finite genus O which contains E. From the previous paragraph we conclude that E is a set of meromorphic approximation. Let us assume, then, that the collection $\{O_j\}$ is infinite. We may assume that each $E \cap O_j$ is non-empty and the collection $\{O_j\}$ is locally finite. Let $\{Q_n\}$ be an exhaustion of R, with $Q_0 = \emptyset$. For $n = 1, 2, \ldots,$ set

$$G_n = \cup\{O_j : O_j \cap Q^{\mathrm{o}}_n \neq \emptyset,\ O_j \cap Q^{\mathrm{o}}_{n-1} = \emptyset\},$$

$$E_n = E \cap G_n, \quad K_n = Q_{n-1} \cup \{E_j : j \leqslant n\}$$

and

$$R_n = Q^{\mathrm{o}}_{n+1} \cup \{G_j : j \leqslant n\}.$$

Note that the sets E_n are disjoint and $\cup E_n = E$. Since the collection $\{O_j\}$ is locally finite but infinite, we may, replacing $\{Q_n\}$ by a subsequence if necessary, assume that each G_n is non-empty. Now, suppose $f \in M(E)$ and $\varepsilon > 0$. Set $f_1 = f$. By Theorem 1, there is a $g_1 \in M(R_1)$ such that $|g_1 - f_1| < \varepsilon 2^{-1}$ on K_1. Define f_2 to be g_1 on $Q_1 \cup E_1$ and to be f on E_2. By Theorem 1, there is a $g_2 \in M(R_2)$ such that $|g_2 - f_2| < \varepsilon 2^{-2}$ on K_2. Continuing in this manner, we construct by induction a sequence $g_n \in M(R_n)$ such that $|g_n - f_n| < \varepsilon 2^{-n}$ on K_n, where f_n is g_{n-1} on $Q_{n-1} \cup \{E_j : j \leqslant n-1\}$ and $f_n = f$ on E_n. The sequence $\{g_n\}$ converges to a function $g \in M(R)$ such that $|g - f| < \varepsilon$ on E. Thus E is a meromorphic approximation set.

We recapitulate: *In an open Riemann surface R, each closed subset E, which is essentially of finite genus, is a meromorphic approximation set.* This result is already known (see [6]).

In a later section, we shall consider an even more general class of sets E, which are sets of meromorphic approximation. This is the class of sets *weakly of infinite genus*, introduced in [16]. It contains the class of sets essentially of finite genus.

It is also worth mentioning that the closed sets with empty interior are meromorphic approximation sets and Roth sets with empty interior are holomorphic approximation sets. See [6,7,12,13].

1. Simultaneous approximation on two sets

1.1. Meromorphic separability

Let A and B be closed subsets of an open Riemann surface R and let f and g be two meromorphic functions on A and B, respectively. We are interested in finding a function h meromorphic on all of R which simultaneously approximates f on A and g on B. If A and B are not disjoint, such a simultaneous approximation is loosely referred to as a fusion of the functions f and g. We shall discuss fusion later.

If A and B are disjoint, a sufficient condition for the possibility of simultaneous approximation on A and B is that their union be a meromorphic approximation set. In this case, we have a Urysohn-type phenomenon in the sense that there exists a global meromorphic function which is close to 0 on one of the two sets and close to 1 on the other. Such a function in some sense "separates" A and B. We now give a precise definition of this concept.

Definition 1. Let A and B be non-empty closed subsets of an open Riemann surface R. If there exists some $g \in M(R)$ such that $g(A)$ and $g(B)$ have disjoint closures then we say that A and B are meromorphically separable.

Lemma 1. *Let A and B be non-empty closed subsets of an open Riemann surface R. The sets A and B are meromorphically separable if and only if for each $\varepsilon > 0$, there exists some $m \in M(R)$ with $\|m\|_A < \varepsilon$ and $\|m - 1\|_B < \varepsilon$.*

Proof. One direction is obvious. Conversely, let g be as in Definition 1. Then $\overline{g(A)}$ and $\overline{g(B)}$ are compact and disjoint subsets of the Riemann sphere. By Runge's theorem, there is some rational function r with $\|r\|_{\overline{g(A)}} < \varepsilon$ and $\|r - 1\|_{\overline{g(B)}} < \varepsilon$. Then $m := r \circ g$ has the desired properties. □

Remarks. Let A and B be disjoint non-empty closed subsets of R.

1. If A and B are compact, or if R is of finite genus, it follows from the Behnke–Stein theorem, or from Theorem 1, that A and B are meromorphically separable. The same is true of course if there exists some open Riemann surface $R' \supset R$ such that the closures of A and B in R' are compact and disjoint.
2. If A and B are meromorphically separable, then there exists some $m \in M_{\mathrm{b}}(A) \cap M_{\mathrm{b}}(B)$ such that $m(A)$ and $m(B)$ have disjoint closures. Indeed, the function in Lemma 1 has the required property.

Question. If the union of two disjoint closed sets is a meromorphic approximation set, then clearly each is an approximation set and they are meromorphically separable. Is the converse true? More precisely, if A and B are meromorphically separable approximation sets, is the union also an approximation set? What if one of the sets is compact?

Suppose E is a meromorphic approximation set in R. We shall introduce in Theorem 2 a condition in terms of certain exhaustions of R, which is sufficient for $E \cup X$ to be a meromorphic approximation set, for each compact set $X \subset R$, and thus sufficient for the meromorphic separability of E and X, when they are disjoint. First, some preliminaries.

For approximation, the notion of a Cauchy differential $\omega(p, q)$ on an open Riemann surface R is fundamental. For each fixed point q in R, $\omega(p, q)$ is a differential in the variable p, whose only pole is a simple pole at q with residue 1. In a local variable for p, we may write

$$\omega(p, q) = g(z, q)\, dz,$$

where g is meromorphic in z. For fixed p, the Cauchy differential $\omega(p, q)$ depends meromorphically on q and its only pole is a simple pole at p in the following sense. For each fixed p and each fixed local variable z at p, with $z(p) = z$, the function $g(z, q)$ is meromorphic in q and its only pole is a simple pole at p. Cauchy differentials were introduced by H. Behnke and K. Stein in their fundamental paper [2] in which they extended Runge's theorem to open Riemann surfaces. The proof of Behnke and Stein that on each open Riemann surface there exists a Cauchy differential is based on the existence of special differentials on compact Riemann surfaces (for which see also [9]). For proofs, without recourse to the theory of compact Riemann surfaces, that on an open Riemann surface there always exists a Cauchy differential, see [12] and [6]. But these approaches use results based on [2]. So they give in fact no alternative way to construct Cauchy differentials, but allow one to recover them using the existence of certain holomorphic functions on open Riemann surfaces.

Definition 2. Let R be an open Riemann surface, $K \subset R$ be compact and $F \subset R$ be closed, with $K \cap F = \emptyset$. We say that (K, F) is a pre-fusion pair if there exist a bounded neighbourhood U of K and a neighbourhood V of F with $\overline{U} \cap \overline{V} = \emptyset$ and $\mathcal{K} := R \setminus (U \cup V)$ compact, a $\mathcal{C}^1$-function $\chi : R \to [0, 1]$ which is identically 1 on U and 0 on V and $\omega(p, q)$ a Cauchy-differential on R such that the positive function C_ω on R given by

$$C_\omega(q) = \iint_{\mathcal{K}} \left|\omega(p, q) \wedge \overline{\partial}\chi(p)\right|,$$

is bounded on F, where $\overline{\partial}\chi$ stands for the differential on R, which in each local coordinate z is given by $(\partial\chi/\partial\overline{z})\, d\overline{z}$.

Alice Roth's fusion lemma [11] is valid on Riemann surfaces in the following version (see [15]).

Lemma 2 (*Fusion lemma*). *Let K and F be disjoint subsets of an open Riemann surface R with K compact and F closed. Then there exists a positive continuous function φ on R with the following property*:

For each compact set $K_0 \subset R$ such that $K \cup F \cup K_0$ is a meromorphic approximation set and all $m_1, m_2 \in M(R)$, there exists some $m \in M(R)$ with

$$|m_1(p) - m(p)| \leqslant \|m_1 - m_2\|_{K_0} \cdot \varphi(p) \quad (p \in K \cup K_0)$$

and

$$|m_2(p) - m(p)| \leqslant \|m_1 - m_2\|_{K_0} \cdot \varphi(p) \quad (p \in F \cup K_0).$$

If (K, F) is a pre-fusion pair, we may take φ to be bounded on F, and thus on $K \cup F \cup K_0$.

Remarks.

1. In the proof of Lemma 2 (see [15]), we may take $\varphi = C_\omega + c$ on $K \cup F \cup K_0$ where $c > 1$ is an arbitrary constant. Thus, the function φ can be chosen bounded on F if and only if the function C_ω in Definition 2 can be.

2. In Definition 2, if F is compact then (K, F) is always a pre-fusion pair since C_ω is continuous on R.
3. Whether (K, F) is a pre-fusion pair depends essentially on the underlying Cauchy differential, which is of course not unique. So it will be a hard question to characterize all pre-fusion pairs on an arbitrary open Riemann surface without detailed knowledge of all Cauchy differentials on it.
4. If (K, F) is a pre-fusion pair, then

   $$a := \sup\{\varphi(p) \,:\, p \in K \cup F \cup K_0\}$$

   is finite, where φ is the function appearing in Lemma 2. The classical form of Alice Roth's fusion lemma [11] in its extended form on open Riemann surfaces is obtained from Lemma 2 if F also is compact (see [6]). But note that compactness of F is not required in Lemma 2.

1.2. Some more meromorphic approximation sets

If we can find some open Riemann surface $R' \supset R$ (inclusion in the sense of conformal embedding), such that E is bounded on R', then E is known to be a meromorphic approximation set. We sketch the proof. The closure of E in R' is compact in R' and hence, by the Behnke–Stein theorem, is a set of approximation in R'. Let $\{Q_n\}_{n\in\mathbb{N}}$ be an exhaustion of R. Since we are able to fulfill the conditions of Definition 2 using an arbitrary Cauchy-differential on R', which also is one on R, and having in mind Remark 1.1 above, we obtain that $(Q_n, E \setminus Q_{n+1})$ is, for all $n \in \mathbb{N}$, a pre-fusion pair. Then, if f is a meromorphic function on E, we can apply the fusion lemma to these pre-fusion pairs to construct a sequence $g_n \in M(R')$ which converges uniformly on Q_k for each $k \in \mathbb{N}$, such that the limit function is meromorphic on $\bigcup_{k\in\mathbb{N}} Q_k = R$ and approximates f on E. This shows that E is a set of meromorphic approximation in R.

Earlier, we indicated that sets essentially of finite genus are always sets of meromorphic approximation. We now consider the more general class of sets E *weakly of infinite genus*, introduced in [16]. We have already mentioned that this class contains the class of sets essentially of finite genus.

For such sets, we can try to modify the previous considerations in the following way: if Q_n is an exhaustion of R by bounded sets, we can try to find a sequence of open Riemann surfaces R_n containing Q_n and a bounded set E_n on R_n, which "almost" equals E. Then, if $g_n \in M(R_n)$ is a sequence and if, for each $k \in \mathbb{N}$, the sequence $(g_n)_{n\geqslant k}$ converges uniformly on Q_k, the limit function is meromorphic on $\bigcup_{k\in\mathbb{N}} Q_k = R$.

Let us now explain the underlying method in greater detail. It is well-known that in any case E can topologically be regarded as a plane set with a number of holes and attached handles. If one removes almost all handles the remaining set is of finite genus. For this set the construction prescribed above can be done and we obtain pre-fusion pairs on a "slightly" modified surface obtained by introducing cuts along appropriate curves *close to the ideal boundary* of R. The corresponding function C_ω (Definition 2) depends essentially on the deleted handles and the cuts along the curves. One can repeat the procedure in this way: we cut off only those handles which are "somewhat closer" to the ideal boundary than in the preceding step. Again we obtain a related function C_ω. If the sequence of all these functions is sufficiently well converging on compact sets of R, one can hope to obtain meromorphic functions on R as local uniform limits of functions, which are constructed on the modified surfaces of the type of R' above. This is the idea behind the concept of the aforementioned class of sets of weakly infinite genus. The definition is given in [16]. It turns out that a sufficiently good behaviour of the functions C_ω can be guaranteed if the handles (belonging to E) “tend” sufficiently fast to the ideal boundary (like a Blaschke sequence to the unit circle). We refer the interested reader to [16].

1.3. Meromorphic approximation on two sets

We were not able to answer the question as to whether the union of two disjoint meromorphic approximation sets is also a meromorphic approximation set. However, if we remove the restriction that the sets be disjoint, the union of two closed meromorphic approximation sets need not be a meromorphic approximation set. In fact, any closed subset E of any Riemann surface R can be decomposed in the form $E = E_1 \cup E_2$, where both E_1 and E_2 are planar (of genus zero) closed subsets of R; hence E_1 and E_2 are meromorphic approximation sets. Recall [7] that there exists a Riemann surface and a closed subset E thereof which is *not* a meromorphic approximation set.

Perhaps the union of two closed meromorphic approximation sets remains a meromorphic approximation set if one of them is compact. The following theorem leads in this direction.

Theorem 2. *Let R be an open Riemann surface and $E \subset R$ be a meromorphic approximation set. Assume that either R is of finite genus or that there exists an exhaustion $\{Q_n\}_{n=1}^{\infty}$ of R by compact sets Q_n such that, for all $n \in \mathbb{N}$, $E \cup Q_n$ is a meromorphic approximation set and $(Q_n, E \setminus Q_{n+1}^{\circ})$ is a pre-fusion pair. Then $E \cup X$ is a meromorphic approximation set for each compact set $X \subset R$.*

Under either of these conditions, it thus follows that E and X are meromorphically separable whenever X is a compact set in $R \setminus E$.

Proof. If R is of finite genus, this follows immediately from Theorem 1.

In the second case, we follow the outline of Alice Roth’s proof of the Localization theorem [11].

We consider the hypothesized exhaustion

$$Q_1 \subset Q_2^{\circ} \subset Q_2 \subset Q_3^{\circ} \subset Q_3 \cdots \subset R$$

and fix some compact set X on R. We may assume that $X \subset Q_1$. Now we define

$$K^n := Q_n, \quad F^n := (E \cup X) \setminus Q_{n+1}^{\circ} = E \setminus Q_{n+1}^{\circ}, \quad K_0^n := (E \cup X) \cap Q_{n+1}.$$

Then $K^n \cup F^n \cup K_0^n = E \cup X \cup Q_n = E \cup Q_n$, and by assumption we know $E \cup Q_n$ is a meromorphic approximation set. So we may apply the fusion lemma, and it gives us a corresponding function φ_n, which is bounded on $E \cup Q_n$. Let

$$a_n := \sup\{\varphi_n(p) \ : \ p \in E \cup Q_n\}.$$

Without loss of generality we may assume that $1 < a_1 < a_2 < \cdots < \infty$.

Moreover let $f \in M(E \cup X)$ be given and fix some positive number ε. We may assume that f has no pole on $E \cup X$. Otherwise we apply Mittag–Leffler’s theorem which is valid on open Riemann surfaces (see [5] or [8]), and find a function $F \in M(R)$, such that $g = f - F$ is holomorphic in a neighbourhood of $E \cup X$. If we succeed to find a function $h \in M(R)$ with $\|g - h\|_{E \cup X} < \varepsilon$, then

$$\|g - h\|_{E \cup X} = \|g + F - (h + F)\|_{E \cup X} = \|f - (h + F)\|_{E \cup X} < \varepsilon.$$

By the Behnke–Stein approximation theorem [2] (Runge's theorem for open Riemann surfaces) we find some $\psi_n \in M(R)$ with

$$|\psi_n(p) - f(p)| < \frac{\varepsilon}{a_n 2^{n+2}} \tag{1}$$

whenever $p \in K_0^n$, and so we conclude that, for all $n \in \mathbb{N}$, we have

$$\begin{aligned} |\psi_{n+1}(p) - \psi_n(p)| &\leqslant |\psi_{n+1}(p) - f(p)| + |\psi_n(p) - f(p)| \\ &< \frac{\varepsilon}{a_{n+1}2^{n+3}} + \frac{\varepsilon}{a_n 2^{n+2}} = \frac{\varepsilon}{a_n 2^{n+2}}\left(1 + \frac{a_n}{2a_{n+1}}\right) < \frac{\varepsilon}{a_n 2^{n+1}} \end{aligned}$$

when $p \in K_0^n$.

The fusion lemma gives us meromorphic functions m_n on R with

$$|m_n(p) - \psi_n(p)| < \frac{\varepsilon}{2^{n+1}} \tag{2}$$

for $p \in K^n \cup K_0^n = Q_n \cup (E \cup X) \cap Q_{n+1}$ and

$$|m_n(p) - \psi_{n+1}(p)| < \frac{\varepsilon}{2^{n+1}} \tag{3}$$

for $p \in F^n \cup K_0^n = E \cup X$. Inequality (2) shows that

$$\sum_{j=n}^{\infty} |m_j(p) - \psi_j(p)| < \varepsilon \sum_{j=n}^{\infty} \frac{1}{2^{j+1}} = \frac{\varepsilon}{2^n}$$

when $p \in Q_n$. Therefore $m(p) := \psi_1(p) + \sum_{j=1}^{\infty} (m_j(p) - \psi_j(p))$ is a meromorphic function on $R = \bigcup_{j=1}^{\infty} Q_n$.

Now we obtain for all $p \in K^1$

$$\begin{aligned} |m(p) - f(p)| &\leqslant |\psi_1(p) - f(p)| + \sum_{j=1}^{\infty} |m_j(p) - \psi_j(p)| \\ &< \frac{\varepsilon}{a_1 2^4} + \sum_{j=1}^{\infty} \frac{\varepsilon}{2^{j+1}} < \varepsilon \end{aligned}$$

by (1) and (2).

Next we take some $p \in (E \cup X) \cap (Q_n \setminus Q_{n-1})$, where $n \geqslant 2$. Then we obtain by (3), (1) and (2)

$$\begin{aligned} &|m(p) - f(p)| \\ &\leqslant \sum_{j=1}^{n-1} |m_j(p) - \psi_{j+1}(p)| + |\psi_n(p) - f(z)| + \sum_{j=n}^{\infty} |m_j(p) - \psi_j(p)| \\ &< \sum_{j=1}^{n-1} \frac{\varepsilon}{2^{j+1}} + \frac{\varepsilon}{a_n 2^{n+3}} + \sum_{j=n}^{\infty} \frac{\varepsilon}{2^{j+1}} = \varepsilon\left(\frac{1}{2} + \frac{1}{a_n 2^{n+3}}\right) < \varepsilon. \end{aligned}$$

Thus we have for all $p \in E \cup X$ the estimation $|m(p) - f(p)| < \varepsilon$. Recalling that $m \in M(R)$, this gives f can be uniformly approximated on $E \cup X$ by functions in $M(R)$ and proves the first part of the theorem.

If $X \cap E = \emptyset$, let W_1 and W_2 be disjoint neighbourhoods of E and X, respectively. The function $f : W_1 \cup W_2 \to X$ defined by $f|_{W_1} \equiv 0$ and $f|_{W_2} \equiv 1$ is in $M(E \cup X)$ and can be uniformly approximated on $E \cup X$ by meromorphic functions on R. Thus E and X are meromorphically separable. □

2. Simultaneous interpolation on an additional set

2.1. Meromorphic interpolation on an additional discrete set

The following is a variant of a famous lemma of Walsh.

Lemma 3. *Let f be a function meromorphic on a compact subset X of the Riemann sphere $\overline{\mathbb{C}}$; let $\mathcal{P}_f$ be the pole set of f on X; let Γ be a finite set of points in X; and let $\mathcal{P}$ be a subset of $\overline{\mathbb{C}}$ which meets every component of $\overline{\mathbb{C}} \setminus X$. There exists a rational function g, all of whose poles lie in $\mathcal{P}_f \cup \mathcal{P}$, which approximates f on X and interpolates f at the points of Γ.*

Proof. If X is the entire Riemann sphere, then f itself is rational and there is nothing to prove. Thus, we assume that X is not the Riemann sphere, and by a Möbius change of variables we may assume that X is a compact subset of the finite complex plane. Let g_o be a rational function having the same poles and principal parts as f on X. Then, $f - g_o$ is holomorphic on X and by an extension of the Walsh lemma [3], there is a rational function h, all of whose poles lie in $\mathcal{P}$, which approximates $f - g_o$ on X and interpolates it at the points of the set Γ. The rational function $g = g_o + h$ has the required properties. □

Lemma 4. *Let R be an open Riemann surface, $E \subset R$ be a holomorphic approximation set and Γ be a closed discrete subset of R disjoint from E. Then for each $\varepsilon > 0$ there is a function $h = h_\varepsilon \in M(R)$ such that $\|h\|_E < \varepsilon$ and h has poles precisely at the points of Γ with, moreover, prescribed principal parts (in some pre-fixed local coordinate systems).*

Proof. Since R is Stein, the additive Cousin problem is solvable on R, so there is an $f \in M(R)$ which has poles precisely at the points of Γ and with prescribed principal parts in some fixed local coordinate system at each of these points. Since $f \in H(E)$ there is a $g \in H(R)$ with $|g - f|$ uniformly small on E. The function $h = g - f$ has the desired properties. □

Lemma 5. *Let R be an open Riemann surface. Suppose we are given a holomorphic approximation set $E \subset R$, a (possibly empty) compact set $K \subset R$,* points $p_1, p_2, \ldots, p_n$ *in $E \cup K$, a set of measure zero $X \subset R$, and a point q not in $E \cup K \cup X$. Moreover let $w \in \overline{\mathbb{C}}$ and $\varepsilon > 0$ be fixed.*

Then for each $\varepsilon > 0$ there exists a function $k = k_\varepsilon \in M(R)$ such that $\|k\|_{E \cup K} < \varepsilon$, $k(q) = w$, $k(p_j) = 0$, $j = 1, \ldots, n$ and k has no poles in X.

Proof. Let h be the function obtained in Lemma 4, where Γ is the singleton $\{q\}$. Let $J = \overline{h(E \cup K)} \cup \{\infty\}$. Using Lemma 3, approximate the function

$$f(p) = \begin{cases} 0 & \text{if } p \in \overline{h(E \cup K)}, \\ w & \text{if } p = \infty \end{cases}$$

by a rational function r which simultaneously interpolates f at the points $h(p_j)$ and at infinity. Let $k = r \circ h$. Then $\|k\|_{E \cup K} < \varepsilon$, $k(q) = w$ and $k(p_j) = 0$, $j = 1, \ldots, n$.

Now we note that $h(X)$ is a subset of the finite complex plane having measure zero. When we invoke Runge's theorem in the proof of Lemma 3, we may choose the poles among any set which meets each complementary component of $\overline{h(E \cup K)}$. Since $h(X)$ has measure zero, it fills no complementary component of $h(E)$ and so we may choose the pole set of the rational function disjoint from $h(X)$. □

Theorem 3. *Suppose E is a meromorphic approximation set in R and $E \neq R$. Then, there exists a point $q \notin E$ such that, for each $f \in M(E)$, for each $\varepsilon > 0$ and for each value $w \in \overline{\mathbb{C}}$, there is a $g \in M(R)$ such that $\|f - g\|_E < \varepsilon$ and $g(q) = w$. Moreover, if $w = \infty$ and a local coordinate is given at q, we may specify the principal part of g at q.*

Proof. If E is a closed discrete subset of R, then the proof is easy.

Suppose now E has an accumulation point p. Choose a point $q_0 \notin E$ and let $f_0 \in H(R \setminus \{q_0\})$ have an essential singularity at q_0. Let g_n be a sequence in $M(R)$ which converges uniformly to f_0 on E. Then g_n is a uniformly Cauchy sequence on E. Suppose $\|g_m - g_n\|_R \leqslant \|g_m - g_n\|_E$ for each m and n. Then, g_n is a uniformly Cauchy sequence on R and hence converges to a function $g \in M(R)$. We have $f_0 = g$ on E and since E has an accumulation point, it follows that $f_0 = g$ on $R \setminus \{q_0\}$. But this is a contradiction, since f_0 has an essential singularity at q_0 and g is meromorphic at q_0. Thus, $\|g_m - g_n\|_R > \|g_m - g_n\|_E$, for infinitely many m and n. Fix such a pair m and n and let $h = g_m - g_n$. Then, for some $q \in R$, we have $|h(q)| > \|h\|_E$. By postcomposing with a rational function we may assume that $\|h\|_E < \varepsilon$ and h takes a prescribed value at q. In case the prescribed value is ∞, we may also specify (in any local coordinate) the principal part of h at q. Now, let $f \in M(E)$ and $w \in \overline{\mathbb{C}}$. There is a $g_0 \in M(R)$ such that $\|f - g_0\|_E < \varepsilon/2$. By the previous discussion, there is an $h \in M(R)$ such that $\|h\|_E < \varepsilon/2$ and $(g_0 + h)(q) = w$. The function $g = g_0 + h$ has the required properties. □

Corollary 1. *Suppose E is a meromorphic approximation set in R and $E \neq R$. Then, there exists a point $q \notin E$ such that $E \cup \{q\}$ is a meromorphic approximation set.*

Proof. Set $w = f(q)$ in the theorem. □

Corollary 2. *A necessary condition that a proper closed subset E of an open Riemann surface R be a meromorphic approximation set is that there exist unbounded functions in $M(R)$, which are bounded on E.*

Proof. Invoke the theorem for f defined to be 0 on E and ∞ at q. □

As a consequence of Lemma 5, we also obtain:

Theorem 4. *Let E be a holomorphic approximation set on an open Riemann surface R; let Γ be a closed discrete set disjoint from E, let X be a set of measure zero on R disjoint from Γ; and let $f \in H(E)$. Fix $\varepsilon > 0$. Then, there is a $g \in M(R)$, having no poles on X, such that $\|g - f\|_E < \varepsilon$ and g assumes prescribed values (finite or infinite) at the points of Γ. In the case of infinite prescribed values at some points of Γ, that is for prescribed poles, we may also prescribe the principal parts (in some fixed local coordinate systems) at these points.*

Proof. We may arrange Γ in a sequence $\{p_j\}_{j=1}^{\infty}$ along with an exhaustion by compact sets K_j of R such that $p_j \in K_{j+1} \setminus K_j$. Let $w_j \in \overline{\mathbb{C}}$ be prescribed values. Let g_0 be a function in

$H(R) \subset M(R)$ satisfying $\|f - g_0\|_E < \varepsilon$. By Lemma 5, there exists a function $g_1 \in M(R)$ such that $\|g_1\|_{E\cup K_1} < \varepsilon/2^2$, g_1 has no poles on X nor on the points p_j, $j > 1$, and such that $g_0 + g_1$ takes the value w_1 at the point p_1. In case $w_1 = \infty$, we may also prescribe the principal part. Using Lemma 5 again, there exists $g_2 \in M(R)$ such that $\|g_2\|_{E\cup K_2} < \varepsilon/2^3$, g_2 has no poles on X nor at the points p_j, $j > 2$, is zero at p_1 and such that $g_0 + g_1 + g_2$ takes the value w_2 at the point p_2, with prescribed principal part if $w_2 = \infty$. Inductively, we construct g_n in $M(R)$ such that $\|g_2\|_{E\cup K_n} < \varepsilon/2^{(n+1)}$, g_n has no poles on X nor at the points p_j, $j > n$, is zero at $p_1, p_2, \ldots, p_{n-1}$ and such that $g_0 + g_1 + \cdots g_n$ takes the value w_n at the point p_n, with prescribed principal part if $w_n = \infty$. The sum $g = \sum_n g_n$ has the required properties. □

Corollary 3. *Let E be a holomorphic approximation set on an open Riemann surface R, and $\Gamma \subset R \setminus E$ be a closed discrete set. Then $E \cup \Gamma$ is a meromorphic approximation set.*

2.2. Holomorphic interpolation on an additional finite set

Theorem 5. *Suppose E is a holomorphic approximation set in R and $E \neq R$. Then, there exists a point $q \notin E$ such that, for each $f \in H(E)$, for each $\varepsilon > 0$ and for each value $w \in \mathbb{C}$, there is an entire function g such that $\|f - g\|_E < \varepsilon$ and $g(q) = w$.*

Proof. The proof is the same as that of Theorem 3, noting that, in the proof, instead of postcomposing with a rational function, we are now allowed to postcompose with a polynomial, since $w \neq \infty$. □

Corollary 4. *Suppose E is a holomorphic approximation set in R and $E \neq R$. Then, there exists a point $q \notin E$ such that $E \cup \{q\}$ is a holomorphic approximation set.*

Recall that if E is a holomorphic approximation set, then E is a Roth set. Analogous to the meromorphic situation, it follows from the corollary that a further necessary condition, that a proper closed subset E of an open Riemann surface R be a holomorphic approximation set is that there exist non-constant entire functions, which are bounded on E. See also Theorem 7 below.

Lemma 6. *Let E be a holomorphic approximation set on an open Riemann surface R and fix $\varepsilon > 0$. If q is not in $\widehat{E}$, then there is an entire function f such that $\|f\|_E < \varepsilon$ and $f(q) = 1$ with multiplicity* 1.

Proof. Suppose that q is not in $\widehat{E}$. Then there is an entire function g such that $|g(q)|$ is larger than $\|g\|_E$. Multiplying by an appropriate complex number, we may assume that $g(q) = 1$ and thus $\|g\|_E$ is strictly less than 1. Now raising g to a sufficiently high power, we obtain an entire function k which is small on E and takes the value 1 at q.

Suppose $k(q)$ assumes the value 1 with multiplicity $m > 1$. By Lemma 4, there is a meromorphic function h which is small on E and has precisely one pole at q of multiplicity $m - 1$. The function $k - 1$ is near -1 on E and still has one pole at q of multiplicity $m - 1$. Thus $f_0 = (k - 1)(h - 1)$ is an entire function, which is near 1 on E and has a zero of multiplicity 1 at q. Finally, the function $f = 1 - f_0$ has the required properties. □

Theorem 6. *Suppose E is a holomorphic approximation set on an open Riemann surface R. Then, it is possible to simultaneously approximate uniformly on E functions in $H(E)$ by entire functions and interpolate arbitrary values at a point q if and only if q is not in $\widehat{E}$.*

Proof. If $q \in \widehat{E}$, then obviously there is no entire function which simultaneously approximates the zero function with an error less than 1 on E and interpolates the value 1 at q.

Suppose now that q is not in $\widehat{E}$. Then, by Lemma 6, there is an entire function which is small on E and takes the value 1 at q. If $w \neq 0$ is a given complex number, and ε is an arbitrarily small positive number, we have in particular the existence of an entire function which assumes the value 1 at q and is less than $\varepsilon/|w|$ in absolute value on E. The function wf then assumes the value w at q and is bounded by ε on E. We have shown that we can interpolate at q using an entire function which is as small as we please on E.

It follows that it is possible to approximate on E and simultaneously interpolate at q by entire functions. Indeed, let h be holomorphic on E, let ε be a positive number and let w be a complex number. By hypothesis, there is an entire function k such that $h-k$ is bounded by $\varepsilon/2$ on E. From the previous paragraph, there is an entire function f which is bounded by $\varepsilon/2$ on E and takes the value $w - k(q)$ at q. The entire function $k + f$ has the required properties. This completes the proof of Theorem 6. □

Corollary 5. *Suppose E is a holomorphic approximation set and q is not in E. Then $E \cup \{q\}$ is a holomorphic approximation set if and only if q is not in $\widehat{E}$.*

Combining Corollary 5 with Corollary 4, we have the following.

Corollary 6. *Suppose E is a holomorphic approximation set and $E \neq R$. Then $\widehat{E} \neq R$.*

Theorem 7. *Let E be a holomorphic approximation set. If $E \neq R$, then there exists an unbounded entire function which is bounded on E.*

Proof. Since the complement of $\widehat{E}$ in R^* is always connected, it follows from the hypothesis, that the complement of E is unbounded. There is an entire function h such that the supremum M_1 of $|h|$ on E is less than the supremum of $|h|$ on R. Let X_j be an exhaustion of R by compact sets. Choose a point q_1 such that $|h(q_1)| > M_1$. Let K_1 be a compact set which contains X_1 and q_1. Let M_2 be the maximum of $|h|$ on K_1. Since h is open we may choose a point q_2 such that $|h(q_2)| > M_2$. Continuing in this manner, we construct an exhaustion K_k and a sequence of points q_k such that q_k lies in K_k but not in the preceding one and $|h(q_k)| > M_k$, where M_k is the maximum of $|h|$ on K_{k-1}. For each k there is a polynomial p_k such that $|p_k| < 2^{-k}$ on the closed disc of radius M_k and

$$\sum_{j=1}^{k} p_j(h(q_k)) = k.$$

Let f_k be the composition h followed by p_k. Then the sum of the f_k is an entire function having the required property. □

Theorem 8. *Suppose E is a holomorphic approximation set. The following assertions are equivalent*:

(1) *for each point q not in E, it is possible to simultaneously approximate uniformly on E by entire functions and interpolate arbitrary values at q*;

(2) $E = \widehat{E}$.

Proof. This is a corollary of Theorem 6. □

2.3. Holomorphic approximation on additional small discs

Let R be a Riemann surface and suppose $z : U \to V$ is a local parameter on R, where U is an open subset of R, V is an open subset of $\mathbb{C}$ and the mapping z is biholomorphic. If Δ is a disc whose closure is contained in V and $D = z^{-1}(\Delta)$, then we say that D is a parametric disc in R and $\overline{D}$ is a closed parametric disc in R.

Theorem 9. *Suppose E is a holomorphic approximation set on an open Riemann surface R and q is a point not in $\widehat{E}$. Then there is a closed parametric disc $\overline{D}$ containing q in its interior and disjoint from E such that $E \cup \overline{D}$ is a holomorphic approximation set.*

Proof. Let q be a point not in $\widehat{E}$. Then, by Lemma 6, there is an entire function h which is bounded by $1/2$ on E and assumes the value 1 at q with multiplicity 1. In particular, there is a closed parametric disc $\overline{D}$ containing q such that $|h - 1| < 1/2$ on $\overline{D}$ and h is schlicht on D.

We now show that $E \cup \overline{D}$ is a holomorphic approximation set. Let f be holomorphic on this union and let $a > 0$. By hypothesis there is an entire function k such that $|k - f| < a$ on E. Let $K \subset \mathbb{C}$ be the union of the closed disc Δ of radius $1/2$ centred at the origin and the closed Jordan domain $h(\overline{D})$. Let φ be the holomorphic function on K defined to be 0 on Δ and to be $(f - k) \circ h^{-1}$ on $h(\overline{D})$. The complement of K in $\mathbb{C}$ is connected and so by Runge's theorem, there is a polynomial p such that $|\varphi - p| < a$ on K. Let ψ be the composition $p \circ h$. Then, $|\psi| < a$ on E and $|\psi - (f - k)| < a$ on $\overline{D}$. Finally, set $g = k + \psi$. Then, $|g - f| < 2a$ on the union of E and $\overline{D}$. This completes the proof. □

Of course, the condition in Theorem 9 is necessary, for if $q \in \widehat{E}$ then there is no such $\overline{D}$, for it would be impossible to approximate, for example, 0 on E and 1 on $\overline{D}$.

In view of the previous results, one might be tempted to think that if it is possible to approximate by entire functions on E, then it is possible to approximate on the union of E with any closed set F disjoint from $\widehat{E}$ on which it is also possible to approximate by entire functions. But this fails already for pairs of compact subsets of plane domains. Indeed, a counterexample is furnished by the Riemann surface consisting of the plane punctured at the origin and two compact subsets which are disjoint concentric circles centred at the origin. The example of the circles, indicates that we must also consider the hull of the union. Of course, if a necessary condition is that F be disjoint from $\widehat{E}$, then by symmetry it is also necessary that E be disjoint from $\widehat{F}$.

Theorem 10. *Suppose $E \subset R$ is a holomorphic approximation set. Then, for each point q not in E, there is a closed parametric disc D containing q and disjoint from E such that $E \cup D$ is a meromorphic approximation set.*

Proof. The proof is the same as that of Theorem 9, using Lemma 4 instead of Lemma 6 and rational approximation instead of polynomial approximation. □

3. Approximation, interpolation and hull notions

3.1. The meromorphic hull

The problem of simultaneous approximation on a set $E \subset R$ and interpolation in some point $q \in R \setminus E$ can also be investigated in terms of a hull as follows. For any closed subset E of an open Riemann surface R we define

$$hull_m(E) := \bigcap_{f \in M_b(E)} f^{-1}(\overline{f(E)})$$

as the meromorphic hull of E. This is the set of all $p \in R$ such that $f(p)$ belongs to the (compact) set $\overline{f(E)}$, for all $f \in M(R)$ which are bounded on E.

Clearly $E \subset hull_m(E)$ and for each $f \in M_b(E)$, we have

$$f(hull_m(E)) \subset \overline{f(E)}.$$

Consequently,

$$M_b(hull_m(E)) = M_b(E) \quad \text{and} \quad hull_m(hull_m(E)) = hull_m(E).$$

If $R = \mathbb{C}$, then $hull_m(E) = E$ for all $E \subset \mathbb{C}$. This is trivial if $E = \mathbb{C}$. If $E \neq \mathbb{C}$ we take some z_0 in the complement of E and consider $f(z) = \frac{1}{z-z_0}$. This meromorphic function is bounded on E and univalent on $\mathbb{C}$. Thus, $f^{-1}(\overline{f(E)}) = E$ and therefore $hull_m(E) = E$.

If R is an arbitrary open Riemann surface this equality is in general not valid. We give an example akin to the surfaces constructed by P. J. Myrberg. Take two copies $\mathbb{D}_1$, $\mathbb{D}_2$ of the unit disk $\mathbb{D}$ and cut off in both the intervals $I_n := [1 - \frac{1}{2n-1}, 1 - \frac{1}{2n}]$ for each $n \in \mathbb{N}$. Then we identify the upper (resp., lower) part of the slit I_n in $\mathbb{D}_1$ with the lower (resp., upper) part of the corresponding slit in $\mathbb{D}_2$. The result is a Riemann surface R_M of infinite genus.

There is a canonical projection $\Pi : R_M \to \mathbb{D}$. For each $w \in \mathbb{D}$ the preimage $\Pi^{-1}(w)$ consists of two points $p_w^1 \in \mathbb{D}_1$ and $p_w^2 \in \mathbb{D}_2$, unless w is an end point of some interval I_n. Let $A := \{p \in R_M : \Re\,\Pi(p) \geqslant 0\}$, a closed subset of R_M.

Now let some $g \in M_b(A)$ be given. We associate the function $g^*(w) := (g(p_w^1) - g(p_w^2))^2$. It is meromorphic on $\mathbb{D}$ and bounded on $\mathbb{D}^+ := \{w \in \mathbb{D} : \Re w \geqslant 0\}$, and $g^*(1 - \frac{1}{n}) = 0$ for all $n \in \mathbb{N}$. Since these zeros are not a Blaschke sequence with respect to $\mathbb{D}^+$, we conclude that $g^* \equiv 0$ on $\mathbb{D}^+$, and thus on $\mathbb{D}$. Thus, $g(p_w^1) = g(p_w^2)$ for all $w \in \mathbb{D}$. This means that each $g \in M_b(A)$ is in fact only a double of a function in $M(\mathbb{D})$, bounded on $\mathbb{D}^+$, which is the same on both unit disks $\mathbb{D}_1$, $\mathbb{D}_2$.

Now we define E as the union of A with some closed portion B_1 of $\mathbb{D}_1$. Then the related portion B_2 of $\mathbb{D}_2$ is automatically contained in $hull_m(E)$, as we see from the considerations above. In fact $hull_m(E) = E \cup \Pi^{-1}(\Pi(B_1))$ (consider the identity on both disks). In particular, if E contains $\mathbb{D}_1$ we obtain $hull_m(E) = R_M$.

Another way to obtain an example for which $hull_m(E) = R$, other than the trivial example $E = R$, is to find an open Riemann surface R having a proper closed subset E for which $M_b(E)$ contains only constant functions. Such an example, again of Myrberg type, was given in [7].

If $E \subset R$ is a set of meromorphic approximation (and $E \neq R$), there always exist unbounded functions in $M_b(E)$ (see Corollary 2 above). This ensures us that $hull_m(E) \neq R$ in this case and we obtain precise information in terms of this hull about the possible locations of the point q mentioned in Theorem 3.

Theorem 11. *Suppose E is a meromorphic approximation set in R and $E \neq R$. Then $hull_m(E) \neq R$, and the following statements are equivalent*:

(i) *For each $f \in M(E)$, for each $\varepsilon > 0$ and for each value $w \in \overline{\mathbb{C}}$, there is a $g \in M(R)$ such that $\|f - g\|_E < \varepsilon$ and $g(q) = w$. Moreover, if $w = \infty$ and a local coordinate is given at q, we may specify the principal part of g at q.*

(ii) *$q \notin hull_m(E)$.*

Proof. We apply (i) with $w = 1$, $f \equiv 0$ and $\varepsilon = \frac{1}{2}$. The resulting function g belongs to $M_{\mathrm{b}}(E)$ and $g(q) = w \notin \overline{g(E)}$. This shows (ii).

If $q \notin hull_m(E)$ we find some $h \in M_{\mathrm{b}}(E)$ with $h(q) \notin \overline{h(E)}$. Applying Runge's theorem (compare the proof of Lemma 1 or of Theorem 3) we may assume that $|h|$ is small on E and takes a prescribed value at q. The remaining arguments are the same as those found at the end of the proof of Theorem 3. □

There is also a relation between the meromorphic hull and the notion of meromorphic separability (Definition 1).

Theorem 12. *Let A and B be closed subsets of an open Riemann surface R. If A and B are meromorphically separable, then*

$$hull_m(A) \cap hull_m(B) = \emptyset.$$

If B is a finite set, then $hull_m(A) \cap hull_m(B) = \emptyset$ implies that A, B are meromorphically separable.

Proof. If A, B are meromorphically separable we find thanks to Lemma 1 some $m \in M(R)$ which approximates 0 on A and 1 on B uniformly. Thus $m \in M_{\mathrm{b}}(A)$ and $m \in M_{\mathrm{b}}(B)$, and we may ensure that

$$\overline{m(A)} \cap \overline{m(B)} = \emptyset.$$

It then follows that the sets $m^{-1}(\overline{m(A)})$ and $m^{-1}(\overline{m(B)})$ are disjoint. This shows that $hull_m(A) \cap hull_m(B) = \emptyset$.

Let $hull_m(A) \cap hull_m(B) = \emptyset$ and B be finite. Then, for each $p \in B$, there exists some $f_p \in M_{\mathrm{b}}(A)$ such that $f_p(p)$ is not in the compact set $\overline{f_p(A)}$. Again by Runge's theorem we may stipulate that $\|f_p\|_A < \frac{1}{2}$ and $f_p(p) = 1$. We can find such a function f_p for each point p of the finite set B. With $f := \prod_{p \in B} f_p$ we obtain $|f(q)| < \frac{1}{2}$ for all $q \in A$ and $f(B) = \{1\}$, which gives the desired conclusion. □

3.2. The holomorphic hull

In analogy with the meromorphic hull we define the holomorphic hull of a closed subset E of the open Riemann surface R as

$$hull_h(E) := \bigcap_{f \in H_{\mathrm{b}}(E)} f^{-1}(\overline{f(E)}).$$

This is the set of all $p \in R$ such that $f(p)$ belongs to the (compact) set $\overline{f(E)}$, for all $f \in H(R)$ which are bounded on E. In this case $|f(p)| \leqslant \|f\|_E$ for all $f \in H_{\mathrm{b}}(E)$. If $g \in H(R) \setminus H_{\mathrm{b}}(E)$ the inequality $g(p) \leqslant \|g\|_E = \infty$ holds trivially. The inclusion $E \subset hull_h(E)$ is obvious. So we have

$$E \subset hull_h(E) \subset \widehat{E}.$$

If we take $R = \mathbb{C}$ and E any compact set, then $hull_h(E) = E$, because the identity is in $H_{\mathrm{b}}(E)$, while $\widehat{E}$ is the union of E and all bounded complementary components.

We refer again to Myrberg's example as explained in the previous subsection. As before we obtain that $hull_h(E) = E \cup \Pi^{-1}(\Pi(B_1))$. If we take B_1 (same notations as above) as a closed annulus in $\mathbb{D} \setminus \mathbb{D}^+$, then we have an example that E can be a proper subset of $hull_h(E)$, and the latter is a proper subset of $\widehat{E}$ as well, using the fact that each $f \in H_b(E) \subset H_b(A)$ is only a double of a holomorphic function on $\mathbb{D}$, which is bounded on $E \supset A$.

Now we investigate the situation if E is a set of holomorphic approximation on an arbitrary open Riemann surface. We have the following.

Theorem 13. *Let E be a closed set of holomorphic approximation on the open Riemann surface R. Then $hull_h(E) = \widehat{E}$.*

Proof. It remains to show that, under these assumptions, $\widehat{E} \subset hull_h(E)$, which means $R \setminus hull_h(E) \subset R \setminus \widehat{E}$.

Let some $q \in R \setminus hull_h(E)$ be given. Then we may take a function $h \in H_b(E)$ with $h(q) \notin \overline{h(E)}$. We assume that $h(q) = 0$; otherwise we replace h by the function $h - h(q)$. Then $a := \inf_E |h| > 0$. The function $g_w(p) := aw/h(p)$ is in $M(R) \cap H(E)$ for each fixed $w \in \mathbb{C}$. So we can find a function $f \in H(R)$ with $\|f - g_w\|_E < \frac{1}{2}$. For all $p \in E$ this implies

$$|f(p)h(p) - aw| = |f(p)h(p) - g_w(p)h(p)| < \frac{\|h\|_E}{2}.$$

Now we take $w = \frac{\|h\|_E}{a}$ and define $\varphi := fh$. This function is holomorphic on R, has a zero in q and fulfills $|\varphi(p) - \|h\|_E| < \frac{\|h\|_E}{2}$ on E. We see that $\varphi(q) = 0$ lies in the complement of the open disk $\{|w - \|h\|_E| < \frac{\|h\|_E}{2}\} =: D$, and D contains $\varphi(E)$. Thus 0 is an inner point of the unbounded component of $\mathbb{C} \setminus \overline{\varphi(E)}$. There exists, by an obvious application of Runge's theorem in the polynomial version, a polynomial ψ with $\|\psi\| < \frac{1}{2}$ on D and $\psi(0) = 1$. The composition $f_o := \psi \circ \varphi$ has the properties:

$$(1)\ f_o \in H(R), \qquad (2)\ \|f_o\|_E < \frac{1}{2}, \qquad (3)\ f_o(q) = 1.$$

This implies that $q \notin \widehat{E}$. □

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